Arnold's Conjectures on Weak Asymptotics and Statistics of Numerical Semigroups $\mathsf{S}(d_1, d_2, d_3)$

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Abstract

Three conjectures #1999–8, #1999–9 and #1999–10 which were posed by V. Arnold [2] and devoted to the statistics of the numerical semigroups are refuted for the case of semigroups generated by three positive integers d_1, d_2, d_3 with $\gcd(d_1, d_2, d_3) = 1$. Weak asymptotics of conductor $C(d_1, d_2, d_3)$ of numerical semigroup and fraction $p(d_1, d_2, d_3)$ of a segment $[0; C(d_1, d_2, d_3) - 1]$ occupied by semigroup are found.

Contents

1 Introduction

Some years ago V. Arnold has posed three conjectures [1], [2], [3] devoted to statistics of numerical semigroups generated by m positive integers $d_1, \ldots, d_m$ for $m \geq 3$. This statistics is concerned with the so–called *weak asymptotics* for the numbers of the integer points in the domains and on the surface in $\mathbb{R}^m$. The conjectures are enumerated in [2] as #1999–8, #1999–9 and #1999–10 and are intimately related to the Frobenius problem for the numerical semigroups where a progress was achieved recently [4] in the case $m = 3$. This is the first nontrivial case where a set of numerical semigroups is separated into symmetric and non–symmetric semigroups with rather different homological properties of their associated polynomial rings [5], [6], [7]. Based on these properties we refute the Arnold's conjectures for semigroups generated by three elements.

The paper is organized in six Sections. In Section 2 we recall the main facts about numerical semigroups generated by three elements and their associated polynomial rings. Following [3], in Section 3 we define a weak asymptotic of numerical functions on semigroups at the typical large vectors. In Section 4 we prove a technical Lemma 2 on statistics of symmetric and non–symmetric semigroups generated by three elements which makes a basis to perform calculations in the following Sections. In Section 5 we refute the conjectures of Arnold for semigroups generated by three elements. In Section 6 we show that two weak asymptotics, for conductor C of semigroup and fraction p of a segment $[0; C-1]$ occupied by semigroup, are not universal and depend on typical vectors where an averaging is performed around. Based on results of [4] we also improve the lower bound of p which was obtained in Section 5 with less powerful methods.

2 Algebra of numerical semigroups $\mathsf{S}\left(d_1, d_2, d_3\right)$

Let $\mathsf{S}\left(d_1, d_2, d_3\right) \subset \mathbb{Z}_+$ be the additive numerical semigroup finitely generated by a minimal set of positive integers $\{d_1, d_2, d_3\}$ such that $d_1 < d_2 < d_3$ and $\gcd(d_1, d_2, d_3) = 1$. It is classically known that $d_1 \geq 3$ [8]. For short we denote the vector (d_1, d_2, d_3) by $\mathbf{d}^3$. The least positive integer (d_1) belonging to $\mathsf{S}\left(\mathbf{d}^3\right)$ is called *the multiplicity*. The smallest integer $C\left(\mathbf{d}^3\right)$ such that all integers $s,\ s \geq C\left(\mathbf{d}^3\right)$, belong to $\mathsf{S}\left(\mathbf{d}^3\right)$ is called *the conductor* of $\mathsf{S}\left(\mathbf{d}^3\right)$,

$$C\left(\mathbf{d}^3\right) := \min\left\{s \in \mathsf{S}\left(\mathbf{d}^3\right) \mid s + \mathbb{Z}_+ \cup \{0\} \subset \mathsf{S}\left(\mathbf{d}^3\right)\right\} . \tag{1}$$

The number $F\left(\mathbf{d}^3\right) = C\left(\mathbf{d}^3\right) - 1$ is referred to as *the Frobenius number*. Denote by $\Delta\left(\mathbf{d}^3\right)$ the complement of $\mathsf{S}\left(\mathbf{d}^3\right)$ in $\mathbb{Z}_+$, i.e. $\Delta\left(\mathbf{d}^3\right) = \mathbb{Z}_+ \setminus \mathsf{S}\left(\mathbf{d}^3\right)$. The cardinalities (#) of the set $\Delta\left(\mathbf{d}^3\right)$ and

the set $\mathsf{S}\left(\mathbf{d}^3\right)\cap[0;F\left(\mathbf{d}^3\right)]$ are called *a number of gaps* $G\left(\mathbf{d}^3\right)$, or *a genus* of $\mathsf{S}\left(\mathbf{d}^3\right)$, and *a number of nongaps* $\widetilde{G}\left(\mathbf{d}^3\right)$, respectively,

$$G\left(\mathbf{d}^3\right):=\#\left\{\Delta\left(\mathbf{d}^3\right)\right\}\ ,\quad \widetilde{G}\left(\mathbf{d}^3\right):=\#\left\{\mathsf{S}\left(\mathbf{d}^3\right)\cap[0;F\left(\mathbf{d}^3\right)]\right\}\ ,\quad \text{so that} \tag{2}$$

$$G\left(\mathbf{d}^3\right)+\widetilde{G}\left(\mathbf{d}^3\right)=C\left(\mathbf{d}^3\right)\ . \tag{3}$$

Notice that two requirements, $\gcd(d_1,d_2,d_3)=1$ and $G\left(\mathbf{d}^3\right)<\infty$, are equivalent.

The semigroup $\mathsf{S}\left(\mathbf{d}^3\right)$ is called *symmetric* iff for any integer s holds

$$s\in\mathsf{S}\left(\mathbf{d}^3\right)\quad\Longleftrightarrow\quad F\left(\mathbf{d}^3\right)-s\notin\mathsf{S}\left(\mathbf{d}^3\right)\ . \tag{4}$$

Otherwise $\mathsf{S}\left(\mathbf{d}^3\right)$ is called *non–symmetric.* The integers $G\left(\mathbf{d}^3\right)$ and $C\left(\mathbf{d}^3\right)$ are related as [9],

$$2G\left(\mathbf{d}^3\right)=C\left(\mathbf{d}^3\right)\ \text{ if }\ \mathsf{S}\left(\mathbf{d}^3\right)\ \text{ is symmetric semigroup, and }\ 2G\left(\mathbf{d}^3\right)>C\left(\mathbf{d}^3\right)\ \text{ otherwise.} \tag{5}$$

Notice that $\mathsf{S}\left(\mathbf{d}^2\right)$ is always symmetric semigroup [10].

Let $R=\mathsf{k}\left[X_1,X_2,X_3\right]$ be a polynomial ring in 3 variables over a field k of characteristic 0 and

$$\pi\ :\quad \mathsf{k}\left[X_1,X_2,X_3\right]\longmapsto\mathsf{k}\left[z^{d_1},z^{d_2},z^{d_3}\right]$$

be the projection induced by $\pi\left(X_i\right)=z^{d_i}$. Denote $\mathsf{k}\left[z^{d_1},z^{d_2},z^{d_m}\right]$ by $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^3\right)\right]$. Then $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^3\right)\right]$ is a graded subring of $\mathsf{k}\left[X_1,X_2,X_3\right]$ and has a presentation as a R–module,

$$\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^3\right)\right]\cong\mathsf{k}\left[X_1,X_2,X_3\right]/\mathcal{I}\left(\mathbf{d}^3\right)\ .$$

The prime ideal $\mathcal{I}\left(\mathbf{d}^3\right)$ is the kernel of the map π and it is minimally generated by a finite number of generators $P_k\left(X_1,X_2,X_3\right)$ such that $\pi\left(P_k\right)=P_k\left(z^{d_1},z^{d_2},z^{d_3}\right)=0$.

The ring $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^3\right)\right]$ is a 1–dim Cohen–Macaulay ring [11], and becomes Gorenstein ring iff $\mathsf{S}\left(\mathbf{d}^3\right)$ is symmetric [12]. Moreover, by [13] the Gorenstein ring $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^3\right)\right]$ is a complete intersection. Denote by $t\left(\mathsf{S}\left(\mathbf{d}^3\right)\right)$ *a type* of the ring $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^3\right)\right]$ which in the case of numerical semigroup coincides with a cardinality of a set $\mathsf{S}'\left(\mathbf{d}^3\right)$ [5], $t\left(\mathsf{S}\left(\mathbf{d}^3\right)\right)=\#\left\{\mathsf{S}'\left(\mathbf{d}^3\right)\right\}$, where

$$\mathsf{S}'\left(\mathbf{d}^3\right)=\left\{x\in\mathbb{Z}\mid x\notin\mathsf{S}\left(\mathbf{d}^3\right),x+s\in\mathsf{S}\left(\mathbf{d}^3\right),\ \text{for all } s\in\mathsf{S}\left(\mathbf{d}^3\right)/\{0\}\right\}.$$

A set $\mathsf{S}'\left(\mathbf{d}^3\right)$ is not empty since $F\left(\mathbf{d}^3\right)\in\mathsf{S}'\left(\mathbf{d}^3\right)$ holds for any minimal generating set (d_1,d_2,d_3).

Henceforth, $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^3\right)\right]$ is a 1–dim local Cohen–Macaulay ring of multiplicity d_1 and type $t\left(\mathsf{S}\left(\mathbf{d}^3\right)\right)$ which satisfies [5]

$$t\left(\mathsf{S}\left(\mathbf{d}^3\right)\right)=\left\{\begin{array}{l}1\ ,\ \ \text{if } \mathsf{S}\left(\mathbf{d}^3\right) \text{ is symmetric}\ ,\\ 2\ ,\ \text{if } \mathsf{S}\left(\mathbf{d}^3\right) \text{ is non-symmetric}\ .\end{array}\right. \tag{6}$$

Theorem 1 and 2 determine important relations between $G\left(\mathbf{d}^3\right)$, $\widetilde{G}\left(\mathbf{d}^3\right)$ and $t\left(\mathsf{S}\left(\mathbf{d}^3\right)\right)$.

Theorem 1 (Theorem 20, [5])

$$G\left(\mathbf{d}^3\right) \le \widetilde{G}\left(\mathbf{d}^3\right) t\left(\mathsf{S}\left(\mathbf{d}^3\right)\right) \ . \tag{7}$$

Theorem 2 (Theorem 2, [6], Corollary at p. 339, [7])

$$G\left(\mathbf{d}^3\right) = \begin{cases} \widetilde{G}\left(\mathbf{d}^3\right) \ , & \textit{iff} \ \ \mathsf{S}\left(\mathbf{d}^3\right) \ \ \textit{is symmetric} \ , \\ 2\widetilde{G}\left(\mathbf{d}^3\right) \ , & \textit{iff} \ \ \mathbf{d}^3 = \{3, 3k+1, 3k+2\} \ , \ \ k \ge 1 \ . \end{cases} \tag{8}$$

The rest of triples $\mathbf{d}^3$ gives rise to non–symmetric semigroups which satisfy a relation [6],

$$G\left(\mathbf{d}^3\right) = 2\widetilde{G}\left(\mathbf{d}^3\right) - u\left(\mathbf{d}^3\right) \ , \quad 1 \le u\left(\mathbf{d}^3\right) < \widetilde{G}\left(\mathbf{d}^3\right) \ . \tag{9}$$

The case $u\left(\mathbf{d}^3\right) = 1$ was studied in [6]: it holds iff $\mathsf{S}\left(\mathbf{d}^3\right)$ is generated by one of two sporadic triples $\mathbf{d}^3 = \{4,5,11\}, \ \{4,7,13\}$ or by one serie, $\mathbf{d}^3 = \{3, 3k+2, 3k+4\}, \ k \ge 1$. As $u\left(\mathbf{d}^3\right)$ increases, the number of sporadic triples climbs significantly. But there are not to our knowledge any general classification of such semigroups. However, it turns out that Theorems 1 and 2 are enough to resolve one of the Arnol'd Conjectures (Conjecture 2, see Section 5.2).

Consider a minimal generating set (d_1, d_2, d_3) and let $g_1 = \gcd(d_2, d_3)$, $g_2 = \gcd(d_3, d_1)$ and $g_3 = \gcd(d_1, d_2)$ be given. We call the semigroup $\overline{\mathsf{S}}\left(d_1, d_2, d_3\right)$,

$$\overline{\mathsf{S}}\left(d_1, d_2, d_3\right) = \mathsf{S}\left(\frac{d_1}{g_2 g_3}, \frac{d_2}{g_1 g_3}, \frac{d_3}{g_1 g_2}\right) \ , \tag{10}$$

the derived semigroup of $\mathsf{S}\left(d_1, d_2, d_3\right)$.

Theorem 3 (Corollary at p. 77, [5]) *The semigroup* $\mathsf{S}\left(d_1, d_2, d_3\right)$ *is symmetric iff its derived semigroup* $\overline{\mathsf{S}}\left(d_1, d_2, d_3\right)$ *is generated by two elements.*

Now we state Theorem about necessary conditions for $\mathsf{S}\left(\mathbf{d}^3\right)$ to be symmetric.

Theorem 4 *If a semigroup* $\mathsf{S}\left(d_1, d_2, d_3\right)$ *is symmetric then its minimal generating set has a following presentation with at least two relatively not prime elements:*

$$\gcd(d_1, d_2) = b \ , \ \ \gcd(d_3, b) = 1 \ , \ \ d_3 \in \mathsf{S}\left(\frac{d_1}{b}, \frac{d_2}{b}\right) \ . \tag{11}$$

Proof Let $\mathsf{S}\left(d_1, d_2, d_3\right)$ be a symmetric semigroup, i.e. $\gcd(d_1, d_2, d_3) = 1$ and (d_1, d_2, d_3) is a minimal generating set. According to Theorem 3 its derived semigroup $\overline{\mathsf{S}}\left(d_1, d_2, d_3\right)$ given in (10) is generated by two elements. Without loss of generality we can put $g_1 = g_2 = 1$ and write,

$$d_3 = c_1 \frac{d_1}{g_3} + c_2 \frac{d_2}{g_3} \ , \quad c_1, c_2 \in \mathbb{Z}_+ \ , \tag{12}$$

that results in

$$\gcd(d_1,d_2)=g_3\ ,\quad \gcd(d_3,g_3)=1\ ,\quad d_3\in \mathsf{S}\left(\frac{d_1}{g_3},\frac{d_2}{g_3}\right)\ . \tag{13}$$

Denoting $g_3=b$ we arrive at (11). $\square$

It appears that (11) gives also efficient conditions for $\mathsf{S}\left(\mathbf{d}^3\right)$ to be symmetric. This follows from Corollary of the early Lemma [14] for semigroup $\mathsf{S}\left(\mathbf{d}^m\right)$

Lemma 1 (Lemma 1, [14]) *Let $\mathsf{S}\left(d_1,\ldots,d_m\right)$ be a numerical semigroup, a and b be positive integers such that: (i) $a\in \mathsf{S}\left(d_1,\ldots,d_m\right)$ and $a\neq d_i$, (ii) $\gcd(a,b)=1$.*
Then a semigroup $\mathsf{S}\left(bd_1,\ldots,bd_m,a\right)$ is symmetric iff $\mathsf{S}\left(d_1,\ldots,d_m\right)$ is symmetric.

Combining Lemma 1 with a fact that a semigroup $\mathsf{S}\left(\mathbf{d}^2\right)$ is always symmetric we arrive at Corollary.

Corollary 1 *Let $\mathsf{S}\left(d_1,d_2\right)$ be a numerical semigroup, a and b be positive integers, $\gcd(a,b)=1$. If $a\in \mathsf{S}\left(d_1,d_2\right)$, then a semigroup $\mathsf{S}\left(bd_1,bd_2,a\right)$ is symmetric.*

In Corollary 1 a requirement $a\neq d_1,d_2$ can be omitted since e.g. a semigroup $\mathsf{S}\left(bd_1,bd_2,d_1\right)$ is generated by two elements (d_1,bd_2) and is also symmetric.

For a sake of completeness finish this Section with efficient and necessary conditions for $\mathsf{S}\left(\mathbf{d}^3\right)$ to be non–symmetric.

Theorem 5 (Theorem 14, [5]) *A semigroup $\mathsf{S}\left(d_1,d_2,d_3\right)$ minimally generated by three pairwise relatively prime elements is non–symmetric.*

Theorem 6 (Corollary at p. 71, [5]) *Let $\mathsf{S}\left(d_1,d_2,d_3\right)$ be a semigroup and $\overline{\mathsf{S}}\left(d_1,d_2,d_3\right)$ be its derived semigroup. Then $\mathsf{S}\left(d_1,d_2,d_3\right)$ and $\overline{\mathsf{S}}\left(d_1,d_2,d_3\right)$ have the same type. In particular, $\mathsf{S}\left(d_1,d_2,d_3\right)$ is non–symmetric iff $\overline{\mathsf{S}}\left(d_1,d_2,d_3\right)$ is non–symmetric.*

More specific details on semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$ will be given in Section 6.1.

3 Weak asymptotics in numerical semigroups $\mathsf{S}\left(\mathbf{d}^m\right)$

Two sequences of real numbers $A(k)$ and $B(k)$, $k\in\mathbb{Z}_+$, are said to have the same *weak asymptotics* [1], or to have the same *growth rate* [3], or to be *Cesáro equivalent* [15], if

$$\lim_{N\to\infty}\frac{\sum_{k=1}^N A(k)}{\sum_{k=1}^N B(k)}=1\ . \tag{14}$$

The limit is weak here: one requires the convergence only for the sums in (14). In the similar way one can consider the Cesáro equivalence of two sequences $A(k)$ and $B(k)$ *at large integers* $k \in \mathbb{Z}_+$. Let us replace k by a neighborhood $\mathbb{U}_{N,r}(k)$ of length $2r$ of a scaled integer $Nk, N \in \mathbb{Z}_+$. Replace the values of $A(k)$ and $B(k)$ by the arithmetic means $A_{N,r}(k)$ and $B_{N,r}(k)$, respectively,

$$A_{N,r}(k) = \frac{1}{2r} \sum_{j=-r}^{r} A(Nk+j) \ , \quad B_{N,r}(k) = \frac{1}{2r} \sum_{j=-r}^{r} B(Nk+j) \ , \quad Nk+j \in \mathbb{U}_{N,r}(k) \ .$$

Two sequences of real numbers $A(k)$ and $B(k)$, $k \in \mathbb{Z}_+$, are said to have the same *weak asymptotics at large* k [15], if

$$\lim_{\substack{r,N\to\infty \\ r(N)/N\to 0}} \frac{A_{N,r}(k)}{B_{N,r}(k)} = 1 \ . \tag{15}$$

A study of weak asymptotics *at the typical large vectors* $\mathbf{d}^m$ for numerical functions $A\left(\mathbf{d}^m\right)$, as conductor or genus of semigroup, over all vectors $\mathbf{b}^m$ comprising a numerical semigroup $\mathsf{S}\left(\mathbf{d}^m\right)$ is much more difficult problem. Call such vectors $\mathbf{b}^m$, $\mathbf{b}^m \in \mathsf{S}\left(\mathbf{d}^m\right)$, *typical.* Arnol'd gave a recept [3] how to average such functions over *the typical large vectors* $\mathbf{b}^m$.

Let $\mathsf{S}\left(\mathbf{d}^m\right)$ be a numerical semigroup, i.e. a generating set $(d_1,\ldots,d_m)$ is minimal. Replace the vector $\mathbf{d}^m$ by a spheric (or cubic) neighborhood $\mathbb{U}_{N,r}\left(\mathbf{d}^m\right)$ of radius r of a scaled vector $N\mathbf{d}^m \in \mathbb{Z}_+^m$, $N \in \mathbb{Z}_+$. Denote by $\mathbf{j}^m$ a vector $(j_1,\ldots,j_m)$. Replace the value $A\left(\mathbf{d}^m\right)$ by the arithmetic mean $A_{N,r}\left(\mathbf{d}^m\right)$ of the functions $A\left(N\mathbf{d}^m+\mathbf{j}^m\right)$ at the vectors $N\mathbf{d}^m+\mathbf{j}^m \in \mathbb{U}_{N,r}\left(\mathbf{d}^m\right)$ whose components $Nd_i+j_i, j_i \in \mathbb{Z}_+$, $-r \le j_i \le r$, satisfy two constraints:

1. The following holds,
$$\gcd(Nd_1+j_1,\ldots,Nd_m+j_m) = 1 \ , \tag{16}$$
otherwise the corresponding numerical semigroup has an infinite complement $\Delta\left(\mathbf{d}^m\right)$.

2. $\{Nd_1+j_1,\ldots,Nd_m+j_m\}$ is *a minimal generating set*, i.e. there are no nonnegative integers $f_{i,k}$ for which a linear dependence holds
$$Nd_i+j_i = \sum_{k\neq i}^{m} f_{i,k}(Nd_k+j_k) \ , \quad f_{i,k} \in \{0,1,\ldots\} \quad \text{for any} \quad i \le m \ , \tag{17}$$
otherwise $N\mathbf{d}^m+\mathbf{j}^m$ does not generate the m–dim numerical semigroup.

We have also choose the averaging radius r and its growth rate such that

$$1 \ll r \ll N \ , \quad r(N)/N \to 0 \quad \text{when} \quad N \to \infty \ . \tag{18}$$

Call the vector $N\mathbf{d}^m + \mathbf{j}^m$ *admissible* if its components satisfy both constraints (16) and (17). Denote by $\mathbb{M}_{N,r}(\mathbf{d}^m)$ an entire set of admissible vectors, $\mathbb{M}_{N,r}(\mathbf{d}^m) \subset \mathbb{U}_{N,r}(\mathbf{d}^m)$,

$$\mathbb{M}_{N,r}(\mathbf{d}^m) = \{N\mathbf{d}^m + \mathbf{j}^m | -r \le j_i \le r, 1 \ll r \ll N, \text{Constraints (16) and (17) are satisfied}\} . \quad (19)$$

Denote by $\#\{\mathbb{M}_{N,r}(\mathbf{d}^m)\}$ a cardinality of $\mathbb{M}_{N,r}(\mathbf{d}^m)$ and notice that $\#\{\mathbb{M}_{N,r}(\mathbf{d}^m)\} < (2r)^m$ since at least $N\mathbf{d}^m \notin \mathbb{M}_{N,r}(\mathbf{d}^m)$ because $\gcd(Nd_1,\ldots,Nd_m) = N$. Write the arithmetic mean,

$$A_{N,r}(\mathbf{d}^m) = \frac{1}{\#\{\mathbb{M}_{N,r}(\mathbf{d}^m)\}} \sum_{j_1,\ldots,j_m=-r}^{r} A(N\mathbf{d}^m + \mathbf{j}^m) , \quad N\mathbf{d}^m + \mathbf{j}^m \in \mathbb{M}_{N,r}(\mathbf{d}^m) . \quad (20)$$

Say that two numerical functions $A(\mathbf{d}^m)$ and $B(\mathbf{d}^m)$ have the same *weak asymptotics at the typical large* $\mathbf{d}^m$ [3], if

$$\lim_{\substack{r,N\to\infty \\ r(N)/N\to 0}} \frac{A_{N,r}(\mathbf{d}^m)}{B_{N,r}(\mathbf{d}^m)} = \lim_{\substack{r,N\to\infty \\ r(N)/N\to 0}} \frac{\sum_{j_1,\ldots,j_m=-r}^{r} A(N\mathbf{d}^m + \mathbf{j}^m)}{\sum_{j_1,\ldots,j_m=-r}^{r} B(N\mathbf{d}^m + \mathbf{j}^m)} = 1 , \quad N\mathbf{d}^m + \mathbf{j}^m \in \mathbb{M}_{N,r}(\mathbf{d}^m) , \quad (21)$$

and denote this equivalence by

$$\mathsf{A}(\mathbf{d}^m) \overset{\substack{asymptotically \\ weak}}{\equiv} \mathsf{B}(\mathbf{d}^m) . \quad (22)$$

4 Statistics of numerical semigroups $\mathsf{S}(N\mathbf{d}^m + \mathbf{j}^m)$, $N \to \infty$

The main difficulties in performing an analytic summation in (20) and (21) are caused by constraints (16) and (17) which are hardly to account for. For this aim let us estimate $\#\{\mathbb{M}_{N,r}(\mathbf{d}^m)\}$ in the limit (18). Represent a set $\mathbb{M}_{N,r}(\mathbf{d}^m)$ as follows,

$$\mathbb{M}_{N,r}(\mathbf{d}^m) = \widehat{\mathbb{M}_{N,r}}(\mathbf{d}^m) \setminus \widetilde{\mathbb{M}_{N,r}}(\mathbf{d}^m) , \quad (23)$$

$$\widehat{\mathbb{M}_{N,r}}(\mathbf{d}^m) = \{N\mathbf{d}^m + \mathbf{j}^m \mid -r \le j_i \le r,\ 1 \ll r \ll N,\ \text{Constraint (16) is satisfied}\} , \quad (24)$$

and a set $\widetilde{\mathbb{M}_{N,r}}(\mathbf{d}^m)$ comprises all vectors $N\mathbf{d}^m + \mathbf{j}^m$ whose generating sets $\{Nd_1 + j_1, \ldots, Nd_m + j_m\}$ are not minimal though they are still satisfying Constraint (16). Consider two sets $\widehat{\mathbb{M}_{N,r}}(\mathbf{d}^m)$ and $\widetilde{\mathbb{M}_{N,r}}(\mathbf{d}^m)$ separately.

Calculate a cardinality of a set $\widehat{\mathbb{M}_{N,r}}(\mathbf{d}^m)$ in the limit (18) by a probabilistic method which dates back to Euler [15]. It is based on a geometric interpretation of probability $\mathcal{P}_{r,\infty}$ that randomly chosen integers $(Nd_1 + j_1, \ldots, Nd_m + j_m)$ from a set $\mathbb{U}_{N,r}(\mathbf{d}^m)$ do not have common divisors

$$\mathcal{P}_{r,\infty} = \lim_{N\to\infty} \frac{\#\left\{\widehat{\mathbb{M}_{N,r}}(\mathbf{d}^m)\right\}}{\#\{\mathbb{U}_{N,r}(\mathbf{d}^m)\}} = \frac{1}{(2r)^m} \lim_{N\to\infty} \#\left\{\widehat{\mathbb{M}_{N,r}}(\mathbf{d}^m)\right\} . \quad (25)$$

Let a tuple $\{Nd_1+j_1,\ldots,Nd_m+j_m\}$ is chosen randomly from a cubic neighborhood $\mathbb{U}_{N,r}\left(\mathbf{d}^m\right)$ of a scaled vector $N\mathbf{d}^m\in\mathbb{Z}_+^m,\ N\in\mathbb{Z}_+$ with the edge length $2r$ such that $1\ll r\ll N$. The least integer which is still contained in $\mathbb{U}_{N,r}\left(\mathbf{d}^m\right)$ is Nd_1-r. Let p be a prime integer such that $p\le Nd_1-r$. A probability that p divides every element Nd_i+j_i in a tuple $\{Nd_1+j_1,\ldots,Nd_m+j_m\}$ is given by p^{-m}. Consequently, $1-p^{-m}$ is a probability that p does not divide any element in this tuple. Multiplying it over all primes such that $p\le Nd_1-r$, we arrive

$$\mathcal{P}_{r,N}=\prod_{2\le p\le Nd_1-r}\left(1-\frac{1}{p^m}\right)\ , \tag{26}$$

where $\mathcal{P}_{r,N}$ gives a probability that integers $(Nd_1+j_1,\ldots,Nd_m+j_m)$, which are randomly chosen from a set $\mathbb{U}_{N,r}\left(\mathbf{d}^m\right)$, do not have common divisors in the range $2\le p\le Nd_1-r$. Taking the limit (18) we get

$$\mathcal{P}_{r,\infty}=\prod_{p}^{\infty}\left(1-\frac{1}{p^m}\right)=\frac{1}{\zeta(m)}\ , \tag{27}$$

where $\zeta(m)$ stands for the Riemann zeta function. The value $\zeta^{-1}(m)$ gives a probability that there are no other integral points on the segment between 0 and an integral point in m–dim space [15]. Its first several values read $\zeta^{-1}(2)=0.6079$, $\zeta^{-1}(3)=0.8319$, $\zeta^{-1}(4)=0.9239$.

Thus, a cardinality of a set $\widehat{\mathbb{M}_{N,r}}\left(\mathbf{d}^m\right)$ can be estimated as

$$\#\left\{\widehat{\mathbb{M}_{N,r}}\left(\mathbf{d}^m\right)\right\}\simeq\frac{(2r)^m}{\zeta(m)}\ . \tag{28}$$

As for a set $\widetilde{\mathbb{M}_{N,r}}\left(\mathbf{d}^m\right)$, the constraint (16) in the case $m=2$ already presumes that $\widehat{\mathbb{M}_{N,r}}\left(\mathbf{d}^2\right)$ comprises all vectors $N\mathbf{d}^2+\mathbf{j}^2$ whose generating sets $\{Nd_1+j_1,Nd_2+j_2\}$ are minimal, and therefore

$$\widetilde{\mathbb{M}_{N,r}}\left(\mathbf{d}^2\right)=\emptyset\ ,\quad \mathbb{M}_{N,r}\left(\mathbf{d}^2\right)\ =\ \widehat{\mathbb{M}_{N,r}}\left(\mathbf{d}^2\right)\ ,\quad \#\left\{\mathbb{M}_{N,r}\left(\mathbf{d}^2\right)\right\}\simeq\frac{4r^2}{\zeta(2)}\ . \tag{29}$$

Calculate a cardinality of a set $\widetilde{\mathbb{M}_{N,r}}\left(\mathbf{d}^m\right)$ in the limit (18) for higher m, $m\ge 3$. Let a vector $N\mathbf{d}^m+\mathbf{j}^m\in\widetilde{\mathbb{M}_{N,r}}\left(\mathbf{d}^m\right)$ be given, i.e. a generating set $\{Nd_1+j_1,\ldots,Nd_m+j_m\}$ is not minimal. According to (17) there exists at least one element Nd_i+j_i which is representable through the rest of the tuple,

$$Nd_i+j_i=\sum_{k\neq i}^{m}f_{i,k}(Nd_k+j_k)\ ,\qquad\text{or}\qquad d_i-\sum_{k\neq i}^{m}f_{i,k}d_k=\frac{1}{N}\left(\sum_{k\neq i}^{m}f_{i,k}j_k-j_i\right)\ , \tag{30}$$

where $f_{i,k}\in\{0,1,\ldots\}$. Taking the limit (18) we get two relations imposed on $N\mathbf{d}^m+\mathbf{j}^m$,

$$d_i=\sum_{k\neq i}^{m}f_{i,k}d_k\ ,\qquad\text{and}\qquad j_i=\sum_{k\neq i}^{m}f_{i,k}j_k\ . \tag{31}$$

The 1st of them claims that the generating set $\{d_1,\ldots,d_m\}$ is not minimal. However, this contradicts an assumption that $\mathsf{S}(\mathbf{d}^m)$ is a numerical semigroup generated by m elements. Thus, a relation (30) can not be satisfied by any choice of $(j_1,\ldots,j_m\}$ and therefore a set $\widetilde{\mathbb{M}_{N,r}}(\mathbf{d}^m)$ in the limit (18) doesn't contain any vectors. Thus, we have

$$\widetilde{\mathbb{M}_{N,r}}(\mathbf{d}^m)=\emptyset\ ,\quad \mathbb{M}_{N,r}(\mathbf{d}^m)\ =\ \widehat{\mathbb{M}_{N,r}}(\mathbf{d}^m)\ ,\quad \#\{\mathbb{M}_{N,r}(\mathbf{d}^m)\}\simeq\frac{(2r)^m}{\zeta(m)}\ . \tag{32}$$

4.1 Statistics of symmetric and non-symmetric semigroups $\mathsf{S}(N\mathbf{d}^3+\mathbf{j}^3)\ ,\ N\to\infty$

In this Section we deal with numerical semigroups $\mathsf{S}\left(N\mathbf{d}^3+\mathbf{j}^3\right)$ generated by three elements only. Consider statistics of symmetric and non–symmetric semigroups $\mathsf{S}\left(N\mathbf{d}^3+\mathbf{j}^3\right)$ corresponding to admissible vectors $N\mathbf{d}^3+\mathbf{j}^3$. Denote by $\mathbb{M}_{N,r}^{sym}\left(\mathbf{d}^3\right)$ and $\mathbb{M}_{N,r}^{nsym}\left(\mathbf{d}^3\right)$ the sets of admissible vectors $N\mathbf{d}^3+\mathbf{j}^3\in\mathbb{M}_{N,r}\left(\mathbf{d}^3\right)$ such that they correspond to the minimal generating sets of symmetric and non–symmetric semigroups, respectively,

$$\begin{aligned}\mathbb{M}_{N,r}^{sym}\left(\mathbf{d}^3\right) &= \left\{N\mathbf{d}^3+\mathbf{j}^3|-r\le j_i\le r, 1\ll r\ll N,\ \mathsf{S}\left(N\mathbf{d}^3+\mathbf{j}^3\right)\ \ \text{is symmetric}\right\}, \\ \mathbb{M}_{N,r}^{nsym}\left(\mathbf{d}^3\right) &= \left\{N\mathbf{d}^3+\mathbf{j}^3|-r\le j_i\le r, 1\ll r\ll N,\ \mathsf{S}\left(N\mathbf{d}^3+\mathbf{j}^3\right)\ \ \text{is non}-\text{symmetric}\right\}.\end{aligned} \tag{33}$$

These sets and their cardinalities ($\#$) are related in the following way,

$$\begin{aligned}\mathbb{M}_{N,r}\left(\mathbf{d}^3\right) &= \mathbb{M}_{N,r}^{sym}\left(\mathbf{d}^3\right)\cup\mathbb{M}_{N,r}^{nsym}\left(\mathbf{d}^3\right)\ , \\ \#\left\{\mathbb{M}_{N,r}\left(\mathbf{d}^3\right)\right\} &= \#\left\{\mathbb{M}_{N,r}^{sym}\left(\mathbf{d}^3\right)\right\}+\#\left\{\mathbb{M}_{N,r}^{nsym}\left(\mathbf{d}^3\right)\right\}\ .\end{aligned} \tag{34}$$

Calculate a cardinality of a set $\mathbb{M}_{N,r}^{sym}\left(\mathbf{d}^3\right)$ in the limit (18) by applying Theorem 4.

Lemma 2 *Let $\mathsf{S}\left(\mathbf{d}^3\right)$ and $\mathsf{S}\left(N\mathbf{d}^3+\mathbf{j}^3\right)$ be numerical semigroups and $\{Nd_1+j_1,Nd_2+j_2,Nd_3+j_3\}$ be a minimal generating set such that*

$$-r\le j_1,j_2,j_3\le r\ ,\ \ 1\ll r\ll N\ . \tag{35}$$

If $r(N)/N\to 0$ when $N\to\infty$ then

$$\lim_{N\to 0}\#\left\{\mathbb{M}_{N,r}^{sym}\left(\mathbf{d}^3\right)\right\}=0\ . \tag{36}$$

Proof Consider a minimal generating set $\{Nd_1+j_1,Nd_2+j_2,Nd_3+j_3\}$ satisfying (35). Suppose that the corresponding semigroup $\mathsf{S}\left(N\mathbf{d}^3+\mathbf{j}^3\right)$ is symmetric. According to Theorem 4 a triple $N\mathbf{d}^3+\mathbf{j}^3$ has necessarily a following presentation:

$$\gcd\left(Nd_1+j_1,Nd_2+j_2\right)=b\ ,\quad b\in\mathbb{Z}_+\ ,\ \ b\ge 2\ , \tag{37}$$

$$\gcd\left(Nd_3+j_3,b\right)=1\ , \tag{38}$$

$$b\left(Nd_3+j_3\right)=c_1\left(Nd_1+j_1\right)+c_2\left(Nd_2+j_2\right)\ ,\quad c_1,c_2\in\mathbb{Z}_+\ . \tag{39}$$

First, consider (37) and find a great common divisor b of integers Nd_1+j_1 and Nd_2+j_2 in the limit (18). Let such b exists, then

$$N d_1 + j_1 = b\, k_1\ , \quad N d_2 + j_2 = b\, k_2\ , \quad k_1, k_2 \in \mathbb{Z}_+\ , \quad \gcd(k_1,k_2) = 1\ , \tag{40}$$

that results in the following

$$k_1(Nd_2+j_2) = k_2(Nd_1+j_1)\ , \quad \rightarrow \quad k_1 d_2 - k_2 d_1 = \frac{1}{N}(k_2 j_1 - k_1 j_2)\ . \tag{41}$$

Taking the limit (18) we get two Diophantine equations

$$k_1 d_2 - k_2 d_1 = 0\ , \qquad k_1 j_2 - k_2 j_1 = 0\ , \tag{42}$$

supplemented by $\gcd(k_1,k_2)=1$. Their solutions read

$$k_1 = \frac{\mathsf{lcm}(d_1,d_2)}{d_2}\ , \quad k_2 = \frac{\mathsf{lcm}(d_1,d_2)}{d_1}\ , \quad j_1 = k_3 d_1\ , \quad j_2 = k_3 d_2\ , \quad k_3 \in \mathbb{Z}_+\ , \quad k_3 \le \frac{r}{d_2}\ . \tag{43}$$

Combining (40) and (43) we get

$$b = (N+k_3)\gcd(d_1,d_2)\ . \tag{44}$$

As for (38), its simple comparison with (44) necessarily claims

$$j_3 \neq k_3 d_3\ . \tag{45}$$

Finally, consider a relation (39) in the limit (18). Similarly to (42) we get two Diophantine equations imposed on the tuples (d_1,d_2,d_3) and (j_1,j_2,j_3) separately,

$$b d_3 = c_1 d_1 + c_2 d_2\ , \qquad \text{and} \qquad b j_3 = c_1 j_1 + c_2 j_2\ . \tag{46}$$

Multiplying the 1st equation by k_3 and making difference between both equations we get

$$b(j_3 - k_3 d_3) = 0\ , \tag{47}$$

that contradicts (45). Thus, a set $\mathbb{M}^{sym}_{N,r}\left(\mathbf{d}^3\right)$ is empty in the limit (18) that proves Lemma. $\square$

Thus, by (32) and (34) we have

$$\#\left\{\mathbb{M}^{nsym}_{N,r}\left(\mathbf{d}^3\right)\right\} \simeq \#\left\{\mathbb{M}_{N,r}\left(\mathbf{d}^3\right)\right\} \simeq \frac{(2r)^3}{\zeta(3)}\ . \tag{48}$$

In other words, a summation in (20) and (21) for $m=3$ is performed over all non–symmetric semigroups $\mathsf{S}\left(N\mathbf{d}^3+\mathbf{j}^3\right)$ exclusively.

5 Arnold's conjectures on weak asymptotics

V. Arnold gave his conjectures on weak asymptotics for the numerical semigroups $\mathsf{S}(\mathbf{d}^m)$ of arbitrary dimension m.

5.1 Conjecture #1999–8 and its discussion

Conjecture #1999–8 deals with the asymptotic behavior of the conductor of the numerical semigroups. We quote from [2] :

Conjecture 1 (#1999–8) *Explore the statistics of $C(\mathbf{d}^m)$ for typical large vectors $\mathbf{d}^m$. Conjecturally,*

$$\mathsf{C}(\mathbf{d}^m) \overset{asymptotically}{\underset{weak}{\equiv}} g_m \sqrt[m-1]{d_1 \cdot \ldots \cdot d_m}\;, \qquad g_m = \sqrt[m-1]{(m-1)!}\;. \tag{49}$$

Define a new function $K_{N,r}(\mathbf{d}^m)$ in the sense of (21),

$$K_{N,r}(\mathbf{d}^m) = \frac{\sum_{j_1,\ldots,j_m=-r}^{r} C(N\mathbf{d}^m + \mathbf{j}^m)}{\sum_{j_1,\ldots,j_m=-r}^{r} \sqrt[m-1]{V(N\mathbf{d}^m + \mathbf{j}^m)}}\;, \quad \text{where} \quad N\mathbf{d}^m + \mathbf{j}^m \in \mathbb{M}_{N,r}(\mathbf{d}^m)\;, \tag{50}$$

and $V(\mathbf{d}^m) = d_1 \cdot \ldots \cdot d_m$. Then Conjecture 1 can been represented as follows,

$$\mathsf{K}(\mathbf{d}^m) = g_m\;, \quad \text{where} \quad \mathsf{K}(\mathbf{d}^m) = \lim_{\substack{r,N\to\infty \\ r(N)/N\to 0}} K_{N,r}(\mathbf{d}^m)\;. \tag{51}$$

For $m = 2$ the corresponding semigroups $\mathsf{S}\left(N\mathbf{d}^2 + \mathbf{j}^2\right)$ are symmetric and the problem is simplified essentially due to the two reasons. First, the constraint (17) is already incorporated into (16). Next, the conductor $C\left(\mathbf{d}^2\right)$ is known due to Sylvester [16], $C(d_1, d_2) = (d_1 - 1)(d_2 - 1)$. Performing the calculation we can verify Conjecture 1 for $m = 2$,

$$\mathsf{K}\left(\mathbf{d}^2\right) = \lim_{\substack{r,N\to\infty \\ r(N)/N\to 0}} \frac{\sum^{r}_{\substack{j_1,j_2=-r \\ \gcd(N\mathbf{d}^2+\mathbf{j}^2)=1}} \left(1 + \frac{j_1-1}{Nd_1}\right)\left(1 + \frac{j_2-1}{Nd_2}\right)}{\sum^{r}_{\substack{j_1,j_2=-r \\ \gcd(N\mathbf{d}^2+\mathbf{j}^2)=1}} \left(1 + \frac{j_1}{Nd_1}\right)\left(1 + \frac{j_2}{Nd_2}\right)} = 1 - \lim_{\substack{r,N\to\infty \\ r(N)/N\to 0}} A_1 + \lim_{\substack{r,N\to\infty \\ r(N)/N\to 0}} A_2\;, \tag{52}$$

where

$$\begin{aligned} A_1 &\simeq \frac{r}{N}\,\frac{d_1^{-1}\sum^{r}_{\substack{j_2=-r \\ \gcd(N\mathbf{d}^2+\mathbf{j}^2)=1}} \left(1 + \frac{j_2}{Nd_2}\right) + d_2^{-1}\sum^{r}_{\substack{j_1=-r \\ \gcd(N\mathbf{d}^2+\mathbf{j}^2)=1}} \left(1 + \frac{j_1}{Nd_2}\right)}{\sum^{r}_{\substack{j_1,j_2=-r \\ \gcd(N\mathbf{d}^2+\mathbf{j}^2)=1}} \left(1 + \frac{j_1}{Nd_1}\right)\left(1 + \frac{j_2}{Nd_2}\right)}\;, \\ A_2 &\simeq \left(\frac{r}{N}\right)^2 \frac{d_1^{-1}d_2^{-1}}{\sum^{r}_{\substack{j_1,j_2=-r \\ \gcd(N\mathbf{d}^2+\mathbf{j}^2)=1}} \left(1 + \frac{j_1}{Nd_1}\right)\left(1 + \frac{j_2}{Nd_2}\right)}\;. \end{aligned}$$

Taking the limit (18) in (52) we have $\mathsf{K}\left(\mathbf{d}^2\right) = 1$.

For $m \geq 3$ the main difficulty in performing an analytic summation in (51) is due to Curtis' theorem [17] on the non–algebraic representation of the Frobenius number $F\left(\mathbf{d}^m\right)$. In other words, $F\left(\mathbf{d}^m\right)$ cannot be expressed by $d_1, \ldots, d_m$ as an algebraic function (see also [18]). In order to overcome this difficulty and discuss Conjecture 1 in the case $m = 3$ we will bound the limit in (51).

Consider the 3–dim version of Conjecture 1 and recall recent results [4] about the lower bounds for conductor in the symmetric and non–symmetric semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$,

$$C\left(\mathbf{d}^3\right) \geq \left\{ \begin{array}{l} \sqrt{3}\sqrt{d_1 d_2 d_3 + 1} - (d_1 + d_2 + d_3) + 1\ ,\ \text{if } \mathsf{S}\left(\mathbf{d}^3\right) \text{ is non-symmetric}\ , \\ 2\sqrt{d_1 d_2 d_3} - (d_1 + d_2 + d_3) + 1\ ,\ \text{if } \mathsf{S}\left(\mathbf{d}^3\right) \text{ is symmetric}\ . \end{array} \right. \tag{53}$$

Define a ratio,

$$v\left(N\mathbf{d}^3 + \mathbf{j}^3\right) = \frac{C\left(N\mathbf{d}^3 + \mathbf{j}^3\right)}{\sqrt{V\left(N\mathbf{d}^3 + \mathbf{j}^3\right)}}\ , \tag{54}$$

and represent $K_{N,r}\left(\mathbf{d}^3\right)$ as follows,

$$K_{N,r}\left(\mathbf{d}^3\right) = \frac{\sum^{r}_{\substack{j_1, j_2, j_3 = -r \\ \gcd(N\mathbf{d}^3 + \mathbf{j}^3) = 1}} v\left(N\mathbf{d}^3 + \mathbf{j}^3\right)\sqrt{V\left(N\mathbf{d}^3 + \mathbf{j}^3\right)}}{\sum^{r}_{\substack{j_1, j_2, j_3 = -r \\ \gcd(N\mathbf{d}^3 + \mathbf{j}^3) = 1}} \sqrt{V\left(N\mathbf{d}^3 + \mathbf{j}^3\right)}}\ . \tag{55}$$

By Lemma 2 a summation in (55) is performed over $N\mathbf{d}^3 + \mathbf{j}^3 \in \mathbb{M}_{N,r}\left(\mathbf{d}^3\right)$ and the corresponding semigroups $\mathsf{S}\left(N\mathbf{d}^3 + \mathbf{j}^3\right)$ are non–symmetric only. A bound (53) is valid for all such admissible vectors. This results in the following,

$$\begin{aligned} v\left(N\mathbf{d}^3 + \mathbf{j}^3\right) \quad \geq \quad & \frac{\sqrt{3}\sqrt{(Nd_1 + j_1)(Nd_2 + j_2)(Nd_3 + j_3) + 1} - N\sum_{i=1}^3 d_i - \sum_{i=1}^3 j_i + 1}{\sqrt{(Nd_1 + j_1)(Nd_2 + j_2)(Nd_3 + j_3)}} > \\ & \sqrt{3} - \frac{d_1 + d_2 + d_3}{\sqrt{N}\sqrt{d_1 d_2 d_3}} \frac{1 + \frac{j_1 + j_2 + j_3 - 1}{N(d_1 + d_2 + d_3)}}{\sqrt{\left(1 + \frac{j_1}{Nd_1}\right)\left(1 + \frac{j_2}{Nd_2}\right)\left(1 + \frac{j_3}{Nd_3}\right)}}\ . \end{aligned} \tag{56}$$

Combining (55) and (56) we get

$$K_{N,r}\left(\mathbf{d}^3\right) > \sqrt{3} - \frac{d_1 + d_2 + d_3}{\sqrt{N}\sqrt{d_1 d_2 d_3}}\left(\frac{\Sigma_1(r, N)}{\Sigma_0(r, N)} + \frac{\Sigma_2(r, N)}{\Sigma_0(r, N)}\right)\ , \tag{57}$$

where

$$\begin{aligned} \Sigma_0(r, N) \quad &= \quad \sum^{r}_{\substack{j_1, j_2, j_3 = -r \\ \gcd(N\mathbf{d}^3 + \mathbf{j}^3) = 1}} \sqrt{V\left(N\mathbf{d}^3 + \mathbf{j}^3\right)} \\ \Sigma_1(r, N) \quad &= \quad \sum^{r}_{\substack{j_1, j_2, j_3 = -r \\ \gcd(N\mathbf{d}^3 + \mathbf{j}^3) = 1}} \sqrt{\frac{V\left(N\mathbf{d}^3 + \mathbf{j}^3\right)}{\left(1 + \frac{j_1}{Nd_1}\right)\left(1 + \frac{j_2}{Nd_2}\right)\left(1 + \frac{j_3}{Nd_3}\right)}}\ , \\ \Sigma_2(r, N) \quad &= \quad \sum^{r}_{\substack{j_1, j_2, j_3 = -r \\ \gcd(N\mathbf{d}^3 + \mathbf{j}^3) = 1}} \frac{j_1 + j_2 + j_3 - 1}{N(d_1 + d_2 + d_3)} \sqrt{\frac{V\left(N\mathbf{d}^3 + \mathbf{j}^3\right)}{\left(1 + \frac{j_1}{Nd_1}\right)\left(1 + \frac{j_2}{Nd_2}\right)\left(1 + \frac{j_3}{Nd_3}\right)}}\ . \end{aligned}$$

Since the indices j_1, j_2, j_3 are runing in the range $[-r, r]$ and $1 \ll r \ll N$ then

$$1 + \frac{j_k}{Nd_k} \geq 1 - \frac{r}{Nd_k} \ , \quad k = 1, 2, 3 \ ,$$

that leads to the following inequalities

$$\frac{\Sigma_1(r,N)}{\Sigma_0(r,N)} \leq \prod_{k=1}^{3} \left(1 - \frac{r}{Nd_k}\right)^{-1/2} , \quad \frac{\Sigma_2(r,N)}{\Sigma_0(r,N)} \leq \frac{3r-1}{N(d_1+d_2+d_3)} \prod_{k=1}^{3} \left(1 - \frac{r}{Nd_k}\right)^{-1/2} . \tag{58}$$

Combining (57) and (58) we obtain

$$K_{N,r}\left(\mathbf{d}^3\right) > \sqrt{3} - \frac{1}{\sqrt{N}} \left(d_1 + d_2 + d_3 + \frac{3r-1}{N}\right) \prod_{k=1}^{3} \left(d_k - \frac{r}{N}\right)^{-1/2} ,$$

and finally the limit yields

$$\mathsf{K}\left(\mathbf{d}^3\right) \geq \sqrt{3} \, . \tag{59}$$

Thus, Conjecture 1 is refuted for $m = 3$.

As for higher dimension, Conjecture 1 for numerical semigroups $\mathsf{S}\left(\mathbf{d}^m\right)$, $m \geq 4$, doesn't contradict the best lower bound for $C(\mathbf{d}^m)$ known today [19]

$$C\left(\mathbf{d}^m\right) \geq g_m \sqrt[m-1]{d_1 \cdot \ldots \cdot d_m} - (d_1 + \ldots + d_m) + 1 \, . \tag{60}$$

This leaves Conjecture 1 open for the case $m \geq 4$.

5.2 Conjecture #1999–9 and its discussion

Conjecture #1999–9 deals with the asymptotic behavior of the average distribution of the numerical semigroup $\mathsf{S}\left(\mathbf{d}^m\right)$ in the interval of integers between 0 and $C\left(\mathbf{d}^m\right)$. Denote by $p\left(\mathbf{d}^m\right)$ a fraction of the segment $[0; C\left(\mathbf{d}^m\right) - 1]$ which is occupied by the semigroup $\mathsf{S}\left(\mathbf{d}^m\right)$,

$$p\left(\mathbf{d}^m\right) = \frac{\widetilde{G}\left(\mathbf{d}^m\right)}{C\left(\mathbf{d}^m\right)} \, . \tag{61}$$

According to (5) the fraction $p\left(\mathbf{d}^3\right)$ satisfies

$$p\left(\mathbf{d}^3\right) = \frac{1}{2} \, , \text{ iff } \mathsf{S}\left(\mathbf{d}^3\right) \text{ is symmetric} \, , \tag{62}$$

$$p\left(\mathbf{d}^3\right) < \frac{1}{2} \, , \text{ iff } \mathsf{S}\left(\mathbf{d}^3\right) \text{ is non–symmetric} \, . \tag{63}$$

Conjecture 2 (#1999–9) *Determine $p\left(\mathbf{d}^m\right)$ for large vectors $\mathbf{d}^m$. Conjecturally, this fraction is asymptotically equal to $1/m$ (with overwhelming probability for large $\mathbf{d}^m$),*

$$\widetilde{\mathsf{G}}\left(\mathbf{d}^m\right) \overset{\substack{asymptotically \\ weak}}{\equiv} \frac{1}{m}\mathsf{C}\left(\mathbf{d}^m\right) \, . \tag{64}$$

The words ' *asymptotically equal* ' and ' *with overwhelming probability for large* $\mathbf{d}^m$ ' presume a weak asymptotics for $p\left(\mathbf{d}^m\right)$ via the averaging procedure decribed in Section 5.1.

Represent Conjecture 2 in the sense of (21),

$$\mathsf{P}\left(\mathbf{d}^m\right) = \lim_{\substack{r,N\to\infty \\ r(N)/N\to 0}} p_{N,r}\left(\mathbf{d}^m\right) = \frac{1}{m}\;, \quad \text{where} \tag{65}$$

$$p_{N,r}\left(\mathbf{d}^m\right) = \frac{\sum_{j_1,\ldots,j_m=-r}^{r} \widetilde{G}\left(N\mathbf{d}^m + \mathbf{j}^m\right)}{\sum_{j_1,\ldots,j_m=-r}^{r} C\left(N\mathbf{d}^m + \mathbf{j}^m\right)}\;, \quad \text{and} \quad N\mathbf{d}^m + \mathbf{j}^m \in \mathbb{M}_{N,r}\left(\mathbf{d}^m\right)\;. \tag{66}$$

For $m = 2$ the corresponding semigroups $\mathsf{S}\left(N\mathbf{d}^2 + \mathbf{j}^2\right)$ are symmetric and by (5) we have,

$$\mathsf{P}\left(\mathbf{d}^2\right) = p_{N,r}\left(\mathbf{d}^2\right) = p\left(N\mathbf{d}^2 + \mathbf{j}^2\right) = p\left(\mathbf{d}^2\right) = \frac{1}{2}\;. \tag{67}$$

Consider the 3–dim version of Conjecture 2 and recall two important results which are worthwhile to discuss Conjecture. First, consider a semigroup $\mathsf{S}\left(\mathbf{d}^3\right)$ and represent $p_{N,r}\left(\mathbf{d}^3\right)$ as follows,

$$p_{N,r}\left(\mathbf{d}^3\right) = \frac{\sum_{j_1,j_2,j_3=-r}^{r} p\left(N\mathbf{d}^3 + \mathbf{j}^3\right) C\left(N\mathbf{d}^3 + \mathbf{j}^3\right)}{\sum_{j_1,j_2,j_3=-r}^{r} C\left(N\mathbf{d}^3 + \mathbf{j}^3\right)}\;. \tag{68}$$

By Lemma 2 a summation in (68) is performed over $N\mathbf{d}^3 + \mathbf{j}^3 \in \mathbb{M}_{N,r}\left(\mathbf{d}^3\right)$ and the corresponding semigroups $\mathsf{S}\left(N\mathbf{d}^3 + \mathbf{j}^3\right)$ are non–symmetric only. Theorems 1 and 2 imply for such admissible vectors the following,

$$p\left(N\mathbf{d}^3 + \mathbf{j}^3\right) = \frac{1}{3}\;, \text{ iff } N\mathbf{d}^3 + \mathbf{j}^3 = \{3, 3k+1, 3k+2\}\;, \quad k \geq 1\;, \tag{69}$$

$$p\left(N\mathbf{d}^3 + \mathbf{j}^3\right) > \frac{1}{3}\;, \text{ otherwise}\;. \tag{70}$$

However (69) doesn't hold for any N, d_1 and j_1 due to (18). Thus, the semigroups $\mathsf{S}\left(N\mathbf{d}^3 + \mathbf{j}^3\right)$ contributing to (68) satisfy (62) and (70), and therefore

$$\frac{1}{3} < p_{N,r}\left(\mathbf{d}^3\right) < \frac{1}{2}\;. \tag{71}$$

Taking the limit $r, N \to \infty$, $r(N)/N \to 0$ in (71) we refute Conjecture 2 for $m = 3$,

$$\frac{1}{3} < \mathsf{P}\left(\mathbf{d}^3\right) < \frac{1}{2}\;. \tag{72}$$

In Section 6 we improve the left hand side of inequality (72) by applying recent results [4] in the Frobenius problem for the numerical semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$.

As for higher dimension, $m \geq 4$, the relations between $G(\mathbf{d}^m)$ and $\widetilde{G}(\mathbf{d}^m)$ do exist [5], [6] and are similar to those given in Theorems 1 and 2,

$$G(\mathbf{d}^m) \leq \widetilde{G}(\mathbf{d}^m)\, t(\mathsf{S}(\mathbf{d}^m)) \ ,$$
$$G(\mathbf{d}^m) = \begin{cases} \widetilde{G}(\mathbf{d}^m) \ , & \text{iff} \ \ \mathsf{S}(\mathbf{d}^m) \ \text{ is symmetric} \ , \\ \widetilde{G}(\mathbf{d}^m)\, t(\mathsf{S}(\mathbf{d}^m)) \ , & \text{iff} \ \ \mathbf{d}^m = \{m, km+1, \ldots, km+m-1\} \ , \ \ k \geq 1 \ . \end{cases}$$

However, the type $t(\mathsf{S}(\mathbf{d}^m))$ in the case $m \geq 4$ doesn't posses such universal properties as in (6). Here there are very mild constraints only,

$$t(\mathsf{S}(\mathbf{d}^m)) = d_1 - 1, \ \text{iff} \ d_1 = m, \ [20] \quad \text{and} \quad t(\mathsf{S}(\mathbf{d}^m)) < d_1 - 1, \ \text{otherwise}, \ [21].$$

These properties are not enough to resolve Conjecture 2 for the case $m \geq 4$ and leave it open meanwhile.

5.3 Conjecture #1999–10 and its discussion

Conjecture #1999–10 deals with the asymptotic behavior of the average distribution of the numerical semigroup $\mathsf{S}(\mathbf{d}^m)$ in the interval of integers between 0 and $C(\mathbf{d}^m)$. Examples show that semigroup $\mathsf{S}(\mathbf{d}^m)$ fills the right half of the segment $[0;\, C(\mathbf{d}^m) - 1]$ more dense.

Conjecture 3 (#1999–10) *Find the typical density of filling the segment* $[0;\, C(\mathbf{d}^m) - 1]$ *asymptotically for large* $\mathbf{d}^m$. *The conjectured behavior of the density* $p_m(s)$ *at a point* $s < C(\mathbf{d}^m)$ *is*

$$p_m(s) = \left(\frac{s}{\mathsf{C}(\mathbf{d}^m)}\right)^{m-1} \ . \tag{73}$$

Such a distribution would immediately imply that the semigroup $\mathsf{S}(\mathbf{d}^m)$ *occupies* $1/m$*–th part of the segment* $[0;\, C(\mathbf{d}^m) - 1]$,

$$\int_0^{\mathsf{C}(\mathbf{d}^m)} p_m(s) ds = \frac{\mathsf{C}(\mathbf{d}^m)}{m} \ . \tag{74}$$

Since Conjecture 3 is strongly related by the last Formula (74) to Conjecture 2 and the latter is refuted for $m = 3$ in Section 5.2 then the conjectured Formula (73) for $p_3(s)$ is not valid.

6 Conjectures #1999–8 and #1999–9 revisited

In Section 5 we have refuted Conjectures 1 and 2 in the case $m = 3$ implicitly but have not found the explicit expressions for $\mathsf{K}(\mathbf{d}^3)$ and $\mathsf{P}(\mathbf{d}^3)$ although both Conjectures ask for them. There is

another point which makes our solutions in Section 5 incomplete. This is an unknown universality of these solutions. In other words, do $\mathsf{K}\left(\mathbf{d}^3\right)$ and $\mathsf{P}\left(\mathbf{d}^3\right)$ depend on the vector $\mathbf{d}^3$ where an averaging is performed around, or they are given by real numbers that is presumed by Arnol'd ? The question remains actual even in the present situation when Conjectures are refuted.

Based on recent results [4] in the Frobenius problem for the numerical semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$ we give in this Section the explicit expressions for $\mathsf{K}\left(\mathbf{d}^3\right)$ and $\mathsf{P}\left(\mathbf{d}^3\right)$ and show that they are not universal. We also improve an inequality (72) by enhancing its lower bound. Before going to the subject we recall recent results [4] in the Frobenius problem for the numerical semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$. We focus on non–symmetric semigroups since by Lemma 2 such semigroups contribute to the values of $\mathsf{K}\left(\mathbf{d}^3\right)$ and $\mathsf{P}\left(\mathbf{d}^3\right)$ only.

6.1 Matrix $\widehat{\mathcal{R}}_3$ of minimal relations, conductor $C\left(\mathbf{d}^3\right)$ and genus $G\left(\mathbf{d}^3\right)$

Let $\mathsf{S}\left(d_1,d_2,d_3\right) \subset \mathbb{Z}_+$ be the additive numerical semigroup finitely generated by a minimal set of positive integers $d_1 < d_2 < d_3$ such that $\gcd(d_1,d_2,d_3) = 1$. Following Johnson [22] define *the minimal relation* for given triple $\mathbf{d}^3 = (d_1,d_2,d_3)$,

$$a_{11}d_1 = a_{12}d_2 + a_{13}d_3\;, \quad a_{22}d_2 = a_{21}d_1 + a_{23}d_3\;, \quad a_{33}d_3 = a_{31}d_1 + a_{32}d_2\;, \quad \text{where} \qquad (75)$$

$$a_{jj} = \min\left\{v_{jj} \mid v_{jj} \geq 2,\; v_{jj}d_j = v_{jk}d_k + v_{jl}d_l,\; v_{jk}, v_{jl} \in \mathbb{Z}_+ \cup \{0\}\right\}\;, \qquad (76)$$

$$\gcd(a_{jj},a_{jk},a_{jl}) = 1\;, \quad \text{and} \quad (j,k,l) = (1,2,3),\; (2,3,1),\; (3,1,2)\;,$$

The uniquely defined values of $v_{ij}, i \neq j$ which give a_{ii} will be denoted by $a_{ij}, i \neq j$. Represent (75) as a matrix equation

$$\widehat{\mathcal{R}}_3 \left(\begin{array}{c} d_1 \\ d_2 \\ d_3 \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right)\;, \quad \widehat{\mathcal{R}}_3 = \left(\begin{array}{rrr} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{array}\right)\;, \quad \left\{\begin{array}{l} \gcd(a_{11},a_{12},a_{13}) = 1 \\ \gcd(a_{21},a_{22},a_{23}) = 1 \\ \gcd(a_{31},a_{32},a_{33}) = 1 \end{array}\right.\;, \qquad (77)$$

and establish *a standard form* of the matrix $\widehat{\mathcal{R}}_3$ satisfying (75) and (76).

For the non–symmetric semigroups the matrix $\widehat{\mathcal{R}}_3$ can be written as follows [22]

$$\widehat{\mathcal{R}}_3 = \left(\begin{array}{ccc} u_1+w_1 & -u_2 & -w_3 \\ -w_1 & u_2+w_2 & -u_3 \\ -u_1 & -w_2 & u_3+w_3 \end{array}\right)\;, \quad \left\{\begin{array}{l} \gcd(u_1,w_2,u_3+w_3) = 1 \\ \gcd(u_2,w_3,u_1+w_1) = 1 \\ \gcd(u_3,w_1,u_2+w_2) = 1\;, \end{array}\right. \qquad (78)$$

where $u_i, w_i \in \mathbb{Z}_+\;,\;\; i = 1,2,3$. The generators d_1, d_2 and d_3 are uniquely defined in the form [22]

$$d_1 = u_2u_3 + w_2w_3 + u_2w_3\;, \quad d_2 = u_3u_1 + w_3w_1 + u_3w_1\;, \quad d_3 = u_1u_2 + w_1w_2 + u_1w_2\;. \qquad (79)$$

The conductor $C\left(\mathbf{d}^3\right)$ and the genus $G\left(\mathbf{d}^3\right)$ are given by [4]

$$C\left(\mathbf{d}^3\right) = 1+\prod_{i=1}^{3}(u_i+w_i)-A_2-B_2-(u_1w_2+u_2w_3+u_3w_1)+max\{A_3,B_3\}\;, \quad (80)$$

$$2G\left(\mathbf{d}^3\right) = 1+\prod_{i=1}^{3}(u_i+w_i)-A_2-B_2-(u_1w_2+u_2w_3+u_3w_1)+A_3+B_3\;,\quad \text{where} \quad (81)$$

$$A_2=u_1u_2+u_3u_1+u_2u_3\;,\quad A_3=u_1u_2u_3\;,\quad B_2=w_1w_2+w_3w_1+w_2w_3\;,\quad B_3=w_1w_2w_3\;.$$

Notice that

$$2G\left(\mathbf{d}^3\right)-C\left(\mathbf{d}^3\right)=min\{A_3,B_3\}\;. \quad (82)$$

6.2 Explicit expression for $\mathsf{K}\left(\mathbf{d}^3\right)$ and its lower bound

In [3] Arnol'd gave a weak version of Conjecture 1:

For growing values of N and r, $r(N)/N\to 0$ when $N\to\infty$, and large $\mathbf{d}^m$ the mean values $C_{N,r}\left(\mathbf{d}^m\right)$ have a limit (probably provided by conjectured formula (49)) which grows as

$$const\;\sqrt[m-1]{d_1\cdot\ldots\cdot d_m}\;. \quad (83)$$

Here a conjectured limit (83) is more weak than (49) since it admits $const\neq g_m$. Although it does claim the similar dependence $\sqrt[m-1]{d_1\cdot\ldots\cdot d_m}$ as Conjecture 1 does. In that sense our solution in Section 5.1 refutes (49) but its weak version (83) remains still open.

Consider a non–symmetric semigroup $\mathsf{S}\left(\mathbf{d}^3\right)$ and calculate a function $K_{N,r}\left(\mathbf{d}^3\right)$ given in (50). Formulas (79) and (80) dictate to perform the averaging of numerical function $A\left(\mathbf{d}^3\right)$ not on the usual 3–dim cubic lattice $\mathbb{Z}_+^3$, where a set $\mathbb{M}_{N,r}\left(\mathbf{d}^3\right)$ is defined by (19), but on the cubic lattice of higher dimension. Namely, denote by $\mathbf{u}^3$ and $\mathbf{w}^3$ two 3–dim tuples (u_1,u_2,u_3) and (w_1,w_2,w_3), respectively. Consider their union $\mathbf{u}^3\cup\mathbf{w}^3=(u_1,u_2,u_3,w_1,w_2,w_3)$ as a tuple in the 6–dim cubic lattice $\mathbb{Z}_+^3\times\mathbb{Z}_+^3$ as follows,

$$\mathbb{Z}_+^3\times\mathbb{Z}_+^3:=\left\{\mathbf{u}^3\cup\mathbf{w}^3\mid \mathbf{u}^3\cup\mathbf{w}^3=(u_1,u_2,u_3,w_1,w_2,w_3)\;,\;u_i,w_i\in\mathbb{Z}_+\right\}\;. \quad (84)$$

A mapping $\mathbb{Z}_+^3\times\mathbb{Z}_+^3\longmapsto\mathbb{Z}_+^3$ is defined by equations (79). In order to find a weak asymptotics replace a scaling in $\mathbb{Z}^3$ lattice, $N^2d_i\in\mathbb{Z}_+,\;N\in\mathbb{Z}_+$, by the scaling in $\mathbb{Z}^3\times\mathbb{Z}_+^3$ lattice, $Nu_i,Nw_i\in\mathbb{Z}_+$, and define a set $\mathbb{A}_{N,r}\left(\mathbf{u}^3\cup\mathbf{w}^3\right)$ on $\mathbb{Z}_+^3\times\mathbb{Z}_+^3$ as follows,

$$\mathbb{A}_{N,r}\left(\mathbf{u}^3\cup\mathbf{w}^3\right)=\left\{\left(N\mathbf{u}^3+\mathbf{j}^3\right)\cup\left(N\mathbf{w}^3+\mathbf{k}^3\right)\;\middle|\;\begin{array}{c}\gcd\left(D_{1,N}(j_i,k_i),D_{2,N}(j_i,k_i),D_{3,N}(j_i,k_i)\right)=1\;,\\ \mathbf{j}^3=(j_1,j_2,j_3)\;,\quad \mathbf{k}^3=(k_1,k_2,k_3)\;,\\ -r\leq j_i,k_i\leq r\;,\;\;1\ll r\ll N\end{array}\right\}$$

where

$$D_{1,N}(j_i,k_i) = (Nu_2+j_2)(Nu_3+j_3)+(Nw_2+k_2)(Nw_3+k_3)+(Nu_2+j_2)(Nw_3+k_3)\ ,$$

$$D_{2,N}(j_i,k_i) = (Nu_3+j_3)(Nu_1+j_1)+(Nw_3+k_3)(Nw_1+k_1)+(Nu_3+j_3)(Nw_1+k_1)\ ,$$

$$D_{3,N}(j_i,k_i) = (Nu_1+j_1)(Nu_2+j_2)+(Nw_1+k_1)(Nw_2+k_2)+(Nu_1+j_1)(Nw_2+k_2)\ .$$

A cardinality of a set $\mathbb{A}_{N,r}\left(\mathbf{u}^3\cup\mathbf{w}^3\right)$ can be estimated in the same way as was done in Section 4.1 for the set $\mathbb{M}_{N,r}\left(\mathbf{d}^3\right)$,

$$\#\left\{\mathbb{A}_{N,r}\left(\mathbf{u}^3\cup\mathbf{w}^3\right)\right\}\simeq\frac{(2r)^6}{\zeta(3)}\ . \tag{85}$$

Substituting (79) and (80) into (50) and averaging over the set $\mathbb{A}_{N,r}\left(\mathbf{u}^3\cup\mathbf{w}^3\right)$ we get

$$K_{N,r}\left(\mathbf{d}^3\right)=\frac{\sum_{\substack{j_1,j_2,j_3\\k_1,k_2,k_3}}^{\mathbb{A}_{N,r}} C_{j_1,j_2,j_3}^{k_1,k_2,k_3}}{\sum_{\substack{j_1,j_2,j_3\\k_1,k_2,k_3}}^{\mathbb{A}_{N,r}}\sqrt{V_{j_1,j_2,j_3}^{k_1,k_2,k_3}}}\ ,\quad\text{where} \tag{86}$$

$$\begin{aligned}
C_{j_1,j_2,j_3}^{k_1,k_2,k_3} \;=\;& \frac{1}{N^3}+\left(u_1+w_1+\frac{j_1+k_1}{N}\right)\left(u_2+w_2+\frac{j_2+k_2}{N}\right)\left(u_3+w_3+\frac{j_3+k_3}{N}\right)+ \\
& \max\left\{\left(u_1+\frac{j_1}{N}\right)\left(u_2+\frac{j_2}{N}\right)\left(u_3+\frac{j_3}{N}\right),\left(w_1+\frac{k_1}{N}\right)\left(w_2+\frac{k_2}{N}\right)\left(w_3+\frac{k_3}{N}\right)\right\}- \\
& \frac{1}{N}\left[\left(u_1+\frac{j_1}{N}\right)\left(u_2+\frac{j_2}{N}\right)+\left(u_3+\frac{j_3}{N}\right)\left(u_1+\frac{j_1}{N}\right)+\left(u_2+\frac{j_2}{N}\right)\left(u_3+\frac{j_3}{N}\right)\right]- \\
& \frac{1}{N}\left[\left(w_1+\frac{k_1}{N}\right)\left(w_2+\frac{k_2}{N}\right)+\left(w_3+\frac{k_3}{N}\right)\left(w_1+\frac{k_1}{N}\right)+\left(w_2+\frac{k_2}{N}\right)\left(w_3+\frac{k_3}{N}\right)\right]- \\
& \frac{1}{N}\left[\left(u_1+\frac{j_1}{N}\right)\left(w_2+\frac{k_2}{N}\right)+\left(u_2+\frac{j_2}{N}\right)\left(w_3+\frac{k_3}{N}\right)+\left(u_3+\frac{j_3}{N}\right)\left(w_1+\frac{k_1}{N}\right)\right]\ ,
\end{aligned} \tag{87}$$

$$\begin{aligned}
V_{j_1,j_2,j_3}^{k_1,k_2,k_3} \;=\;& \left[\left(u_2+\frac{j_2}{N}\right)\left(u_3+\frac{j_3}{N}\right)+\left(w_2+\frac{k_2}{N}\right)\left(w_3+\frac{k_3}{N}\right)+\left(u_2+\frac{j_2}{N}\right)\left(w_3+\frac{k_3}{N}\right)\right]\times \\
& \left[\left(u_3+\frac{j_3}{N}\right)\left(u_1+\frac{j_1}{N}\right)+\left(w_3+\frac{k_3}{N}\right)\left(w_1+\frac{k_1}{N}\right)+\left(u_3+\frac{j_3}{N}\right)\left(w_1+\frac{k_1}{N}\right)\right]\times \\
& \left[\left(u_1+\frac{j_1}{N}\right)\left(u_2+\frac{j_2}{N}\right)+\left(w_1+\frac{k_1}{N}\right)\left(w_2+\frac{k_2}{N}\right)+\left(u_1+\frac{j_1}{N}\right)\left(w_2+\frac{k_2}{N}\right)\right]\ .
\end{aligned}$$

An upper limit in (86) means that a summation is performed for $\left(N\mathbf{u}^3+\mathbf{j}^3\right)\cup\left(N\mathbf{w}^3+\mathbf{k}^3\right)\in\mathbb{A}_{N,r}\left(\mathbf{u}^3\cup\mathbf{w}^3\right)$. Bearing in mind that $\sum_{\substack{j_1,j_2,j_3\\k_1,k_2,k_3}}^{\mathbb{A}_{N,r}} 1=\#\left\{\mathbb{A}_{N,r}\left(\mathbf{u}^3\cup\mathbf{w}^3\right)\right\}$ and estimating the terms,

$$\left|\sum_{\substack{j_1,j_2,j_3\\k_1,k_2,k_3}}^{\mathbb{A}_{N,r}}\frac{j_i}{N}\right|\simeq\left|\sum_{\substack{j_1,j_2,j_3\\k_1,k_2,k_3}}^{\mathbb{A}_{N,r}}\frac{k_i}{N}\right|<\frac{2r}{N}(2r)^6,\quad\left|\sum_{\substack{j_1,j_2,j_3\\k_1,k_2,k_3}}^{\mathbb{A}_{N,r}}\frac{j_ij_l}{N^2}\right|\simeq\left|\sum_{\substack{j_1,j_2,j_3\\k_1,k_2,k_3}}^{\mathbb{A}_{N,r}}\frac{j_ik_l}{N^2}\right|\simeq\left|\sum_{\substack{j_1,j_2,j_3\\k_1,k_2,k_3}}^{\mathbb{A}_{N,r}}\frac{k_ik_l}{N^2}\right|<\frac{(2r)^2}{N^2}(2r)^6,$$

$$\left|\sum_{\substack{j_1,j_2,j_3\\k_1,k_2,k_3}}^{\mathbb{A}_{N,r}}\frac{j_ij_lj_n}{N^3}\right|\simeq\left|\sum_{\substack{j_1,j_2,j_3\\k_1,k_2,k_3}}^{\mathbb{A}_{N,r}}\frac{j_ij_lk_n}{N^3}\right|\simeq\left|\sum_{\substack{j_1,j_2,j_3\\k_1,k_2,k_3}}^{\mathbb{A}_{N,r}}\frac{j_ik_lk_n}{N^3}\right|\simeq\left|\sum_{\substack{j_1,j_2,j_3\\k_1,k_2,k_3}}^{\mathbb{A}_{N,r}}\frac{k_ik_lk_n}{N^3}\right|<\frac{(2r)^3}{N^3}(2r)^6\ ,\quad\text{etc,}$$

we arrive in accordance with (85) to the leading terms $\mathsf{K}_{\mathsf{n}}(u_i,w_i)$ and $\mathsf{K}_{\mathsf{d}}(u_i,w_i)$ which are contributing to the both sums in (86) in the limit $r,N\to\infty$, $r(N)/N\to 0$,

$$\frac{\sum_{\substack{j_1,j_2,j_3\\ k_1,k_2,k_3}}^{\mathbb{A}_{N,r}} C_{j_1,j_2,j_3}^{k_1,k_2,k_3}}{\#\left\{\mathbb{A}_{N,r}\left(\mathbf{u}^3\cup\mathbf{w}^3\right)\right\}}\simeq \mathsf{K}_{\mathsf{n}}(u_i,w_i)+\mathcal{O}\left(\frac{r}{N}\right),\quad \frac{\sum_{\substack{j_1,j_2,j_3\\ k_1,k_2,k_3}}^{\mathbb{A}_{N,r}} \sqrt{V_{j_1,j_2,j_3}^{k_1,k_2,k_3}}}{\#\left\{\mathbb{A}_{N,r}\left(\mathbf{u}^3\cup\mathbf{w}^3\right)\right\}}\simeq \mathsf{K}_{\mathsf{d}}(u_i,w_i)+\mathcal{O}\left(\frac{r}{N}\right),\quad (88)$$

where

$$\begin{aligned}\mathsf{K}_{\mathsf{n}}(u_i,w_i) &= (u_1+w_1)(u_2+w_2)(u_3+w_3)+max\{u_1u_2u_3,w_1w_2w_3\}\ ,\\ \mathsf{K}_{\mathsf{d}}(u_i,w_i) &= \sqrt{(u_2u_3+w_2w_3+u_2w_3)(u_3u_1+w_3w_1+u_3w_1)(u_1u_2+w_1w_2+u_1w_2)}\ .\end{aligned}$$

Finally, we obtain the expression for $\mathsf{K}\left(\mathbf{d}^3\right)$ in accordance with (51)

$$\mathsf{K}\left(\mathbf{d}^3\right)=\frac{(u_1+w_1)(u_2+w_2)(u_3+w_3)+max\{u_1u_2u_3,w_1w_2w_3\}}{\sqrt{(u_2u_3+w_2w_3+u_2w_3)(u_3u_1+w_3w_1+u_3w_1)(u_1u_2+w_1w_2+u_1w_2)}}\ .\quad (89)$$

One can show that $\mathsf{K}\left(\mathbf{d}^3\right)$ attains its minimal value, $\mathsf{K}\left(\mathbf{d}^3\right)=\sqrt{3}$ when $u_1=w_1$, $u_2=w_2$ and $u_3=w_3$ (see Appendix A). This nicely concides with (59). The representations (89) tells one more important thing: $\mathsf{K}\left(\mathbf{d}^3\right)$ is not universal and depends on the vector $\mathbf{d}^3$ where an averaging is performed around. This refutes Conjecture 1 in its weak version (83).

6.3 Explicit expression for $\mathsf{P}\left(\mathbf{d}^3\right)$ and its lower bound

Consider the non–symmetric semigroup $\mathsf{S}\left(\mathbf{d}^3\right)$ and define a new ratio,

$$q\left(\mathbf{d}^3\right):=\frac{G\left(\mathbf{d}^3\right)-\widetilde{G}\left(\mathbf{d}^3\right)}{C\left(\mathbf{d}^3\right)}\ .\quad (90)$$

Associate with it a corresponding function $q_{N,r}\left(\mathbf{d}^m\right)$,

$$q_{N,r}\left(\mathbf{d}^m\right)=\frac{\sum_{j_1,\ldots,j_m=-r}^{r}\left(G\left(N\mathbf{d}^m+\mathbf{j}^m\right)-\widetilde{G}\left(N\mathbf{d}^m+\mathbf{j}^m\right)\right)}{\sum_{j_1,\ldots,j_m=-r}^{r}C\left(N\mathbf{d}^m+\mathbf{j}^m\right)}\ ,\quad \text{where}\quad N\mathbf{d}^m+\mathbf{j}^m\in\mathbb{M}_{N,r}\left(\mathbf{d}^m\right).$$

Both fractions, $p\left(\mathbf{d}^3\right)$ and $q\left(\mathbf{d}^3\right)$, are readily related to each other,

$$p\left(\mathbf{d}^3\right)=\frac{1}{2}\left(1-q\left(\mathbf{d}^3\right)\right)\ .\quad (91)$$

The same relation holds for theier weak asymptotics,

$$\mathsf{P}\left(\mathbf{d}^3\right)=\frac{1}{2}\left(1-\mathsf{Q}\left(\mathbf{d}^3\right)\right)\ ,\quad \text{where}\quad \mathsf{Q}\left(\mathbf{d}^3\right)=\lim_{\substack{r,N\to\infty\\ r(N)/N\to 0}} q_{N,r}\left(\mathbf{d}^m\right)\ .\quad (92)$$

Perform the averaging of $q_{N,r}\left(\mathbf{d}^3\right)$ over the set $\mathbb{A}_{N,r}\left(\mathbf{u}^3\cup\mathbf{w}^3\right)$ on the 6–dim cubic lattice $\mathbb{Z}_+^3\times\mathbb{Z}_+^3$ in terms of the $\widehat{\mathcal{R}}_3$ matrix entries in the same way as was done in Section 6.2 for $K_{N,r}\left(\mathbf{d}^3\right)$. Bearing in mind (82) we have,

$$q_{N,r}\left(\mathbf{d}^m\right)=\frac{\sum_{\substack{j_1,j_2,j_3\\k_1,k_2,k_3}}^{\mathbb{A}_{N,r}} M_{j_1,j_2,j_3}^{k_1,k_2,k_3}}{\sum_{\substack{j_1,j_2,j_3\\k_1,k_2,k_3}}^{\mathbb{A}_{N,r}} C_{j_1,j_2,j_3}^{k_1,k_2,k_3}}\;,\quad\text{where}\quad \left(N\mathbf{u}^3+\mathbf{j}^3\right)\cup\left(N\mathbf{w}^3+\mathbf{k}^3\right)\in\mathbb{A}_{N,r}\left(\mathbf{u}^3\cup\mathbf{w}^3\right). \tag{93}$$

A denominator of (93) is defined in (87), and a numerator reads,

$$M_{j_1,j_2,j_3}^{k_1,k_2,k_3}=\min\left\{\left(u_1+\frac{j_1}{N}\right)\left(u_2+\frac{j_2}{N}\right)\left(u_3+\frac{j_3}{N}\right),\left(w_1+\frac{k_1}{N}\right)\left(w_2+\frac{k_2}{N}\right)\left(w_3+\frac{k_3}{N}\right)\right\}\;.$$

Performing summation in (93) by applying the similar considerations as in Section 6.2 and taking the limit $r,N\to\infty$, $r(N)/N\to 0$ we get finally,

$$\mathsf{Q}\left(\mathbf{d}^3\right)=\frac{min\{u_1u_2u_3,w_1w_2w_3\}}{(u_1+w_1)(u_2+w_2)(u_3+w_3)+\max\{u_1u_2u_3,w_1w_2w_3\}}\;. \tag{94}$$

Represent (94) as follows,

$$\mathsf{Q}\left(\mathbf{d}^3\right)=\frac{min\{1,\rho_1\rho_2\rho_3\}}{(1+\rho_1)(1+\rho_2)(1+\rho_3)+max\{1,\rho_1\rho_2\rho_3\}}\;,\quad \rho_i=\frac{u_i}{w_i}\;,\quad 0<\rho_i<\infty\;.$$

Making use of inequalities [23] for three basic polynomial invariants $\Gamma_1=\rho_1+\rho_2+\rho_3$, $\Gamma_2=\rho_1\rho_2+\rho_2\rho_3+\rho_3\rho_1$ and $\Gamma_3=\rho_1\rho_2\rho_3$ of symmetric group S_3 acting on the set $\{\rho_1,\rho_2,\rho_3\}$,

$$\Gamma_1\geq\sqrt{3\Gamma_2}\geq 3\sqrt[3]{\Gamma_3}\;,$$

we get $\mathsf{Q}\left(\mathbf{d}^3\right)>0$ and

$$\mathsf{Q}\left(\mathbf{d}^3\right)\;=\;\frac{1}{1+\Gamma_1+\Gamma_2+2\Gamma_3}<\frac{1}{9}\;,\quad\text{if}\quad\Gamma_3\geq 1\;, \tag{95}$$

$$\mathsf{Q}\left(\mathbf{d}^3\right)\;=\;\frac{\Gamma_3}{2+\Gamma_1+\Gamma_2+\Gamma_3}=\frac{1}{1+\Gamma_1\Gamma_3^{-1}+\Gamma_2\Gamma_3^{-1}+2\Gamma_3^{-1}}<\frac{1}{9}\;,\quad\text{if}\quad\Gamma_3\leq 1\;, \tag{96}$$

since $\Gamma_2\Gamma_3^{-1}=\rho_1^{-1}+\rho_2^{-1}+\rho_3^{-1}$, $\Gamma_1\Gamma_3^{-1}=\rho_1^{-1}\rho_2^{-1}+\rho_2^{-1}\rho_3^{-1}+\rho_3^{-1}\rho_1^{-1}$ and $\Gamma_3^{-1}=\rho_1^{-1}\rho_2^{-1}\rho_3^{-1}$ can be considered as basic polynomial invariants of symmetric group S_3 acting on the set $\{\rho_1^{-1},\rho_2^{-1},\rho_3^{-1}\}$.

The case $\rho_1=\rho_2=\rho_3=1$ has to be excluded since the corresponding matrix of minimal relations has the entries $u_1=w_1$, $u_2=w_2$ and $u_3=w_3$ that results in $\gcd(d_1,d_2,d_3)=3$. This is why both inequalities in (95) and (96) are rigorous. Finally, we obtain by (92) the lower and upper bounds for $\mathsf{P}\left(\mathbf{d}^3\right)$,

$$\frac{4}{9}<\mathsf{P}\left(\mathbf{d}^3\right)<\frac{1}{2}\;. \tag{97}$$

The representations (94) claims that $\mathsf{P}\left(\mathbf{d}^3\right)$ is not universal and depends on the vector $\mathbf{d}^3$ where an averaging is performed around.

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A Lower bound of $\mathsf{K}\left(\mathbf{d}^3\right)$

Represent the function $\mathsf{K}\left(\mathbf{d}^3\right)$ given in (89) as follows,

$$\mathsf{K}\left(\mathbf{d}^3\right)=\frac{(1+\rho_1)(1+\rho_2)(1+\rho_3)+max\{1,\rho_1\rho_2\rho_3\}}{\sqrt{(1+\rho_2\rho_3+\rho_2)(1+\rho_3\rho_1+\rho_3)(1+\rho_1\rho_2+\rho_1)}}\;,\quad \rho_i=\frac{u_i}{w_i}\;,\quad 0<\rho_i<\infty\;, \tag{A1}$$

and consider its square, $\mathsf{K}^2\left(\mathbf{d}^3\right)=\mathsf{L}\left(\rho_1,\rho_2,\rho_3\right)$.

First, prove that $\mathsf{L}\left(\rho_1,\rho_2,\rho_3\right)$ is unbounded from above. Consider $0<\overline{\rho_i}<\infty$ such that $\overline{\rho_1}=1/(\overline{\rho_2}\;\overline{\rho_3})$ and $\overline{\rho_2},\overline{\rho_3}\gg 1,\overline{\rho_1}\ll 1$. Calculate a leading term in $\mathsf{L}\left(\overline{\rho_1},\overline{\rho_2},\overline{\rho_3}\right)$

$$\mathsf{L}\left(\overline{\rho_1},\overline{\rho_2},\overline{\rho_3}\right)=\frac{[(1+\overline{\rho_1})(1+\overline{\rho_2})(1+\overline{\rho_3})+1\}]^2}{(1+\overline{\rho_2}\;\overline{\rho_3}+\overline{\rho_2})(1+\overline{\rho_3}\;\overline{\rho_1}+\overline{\rho_3})(1+\overline{\rho_1}\;\overline{\rho_2}+\overline{\rho_1})}\simeq\frac{\overline{\rho_2}^2\overline{\rho_3}^2}{\overline{\rho_2}^2\overline{\rho_3}^2\overline{\rho_1}}=\frac{1}{\overline{\rho_1}}\gg 1\;. \tag{A2}$$

The last inequality proves a statement.

Observe that $\mathsf{L}\left(\rho_1,\rho_2,\rho_3\right)$ is invariant under cyclic permutation of variables ρ_1,ρ_2,ρ_3,

$$\mathsf{L}\left(\rho_1,\rho_2,\rho_3\right)=\mathsf{L}\left(\rho_2,\rho_3,\rho_1\right)=\mathsf{L}\left(\rho_3,\rho_1,\rho_2\right)\;, \tag{A3}$$

and can be represented in four polynomial invariants Γ_i of the cyclic group C_3 [24],

$$\Gamma_1=\rho_1+\rho_2+\rho_3\;,\quad \Gamma_2=\rho_1\rho_2+\rho_2\rho_3+\rho_3\rho_1\;,\quad \Gamma_3=\rho_1\rho_2\rho_3\;, \tag{A4}$$
$$\Gamma_4=(\rho_1-\rho_2)(\rho_2-\rho_3)(\rho_3-\rho_1)\;,\quad \text{where}\;\;\Gamma_4^2=\Gamma_1^2\Gamma_2^2+18\Gamma_1\Gamma_2\Gamma_3-4\Gamma_2^3-4\Gamma_1^3\Gamma_3-27\Gamma_3^2\;.$$

In both regions, $\Gamma_3>1$ and $\Gamma_3<1$, the function $\mathsf{L}\left(\rho_1,\rho_2,\rho_3\right)$ is differentiable and attains its extremal values if $\partial\mathsf{L}/\partial\rho_i=0,\;i=1,2,3$. In other words,

$$\frac{\partial\mathsf{L}}{\partial\rho_i}=\sum_{j=1}^4\frac{\partial\mathsf{L}}{\partial\Gamma_j}\frac{\partial\Gamma_j}{\partial\rho_i}=0\;,\quad i=1,2,3\;,$$

or

$$\left(\frac{\partial\mathsf{L}}{\partial\Gamma_1}+\frac{\partial\mathsf{L}}{\partial\Gamma_4}\frac{\partial\Gamma_4}{\partial\Gamma_1}\right)\frac{\partial\Gamma_1}{\partial\rho_i}+\left(\frac{\partial\mathsf{L}}{\partial\Gamma_2}+\frac{\partial\mathsf{L}}{\partial\Gamma_4}\frac{\partial\Gamma_4}{\partial\Gamma_2}\right)\frac{\partial\Gamma_2}{\partial\rho_i}+\left(\frac{\partial\mathsf{L}}{\partial\Gamma_3}+\frac{\partial\mathsf{L}}{\partial\Gamma_4}\frac{\partial\Gamma_4}{\partial\Gamma_3}\right)\frac{\partial\Gamma_3}{\partial\rho_i}=0\;. \tag{A5}$$

Substituting (A4) into (A5) and removing singular multiplier $1/\Gamma_4$ after taking derivatives $\partial\Gamma_4/\partial\Gamma_i$ we get three equations for $i=1,2,3$,

$$\left(\Gamma_4\frac{\partial\mathsf{L}}{\partial\Gamma_1}+K_{41}\frac{\partial\mathsf{L}}{\partial\Gamma_4}\right)\frac{\partial\Gamma_1}{\partial\rho_i}+\left(\Gamma_4\frac{\partial\mathsf{L}}{\partial\Gamma_2}+K_{42}\frac{\partial\mathsf{L}}{\partial\Gamma_4}\right)\frac{\partial\Gamma_2}{\partial\rho_i}+\left(\Gamma_4\frac{\partial\mathsf{L}}{\partial\Gamma_3}+K_{43}\frac{\partial\mathsf{L}}{\partial\Gamma_4}\right)\frac{\partial\Gamma_3}{\partial\rho_i}=0\;, \tag{A6}$$

where

$$K_{41}=\Gamma_1\Gamma_2^2+9\Gamma_2\Gamma_3-6\Gamma_1^2\Gamma_3\ ,\quad K_{42}=\Gamma_1^2\Gamma_2+9\Gamma_1\Gamma_3-6\Gamma_2^2\ ,\quad K_{43}=9\Gamma_1\Gamma_2-2\Gamma_1^3-27\Gamma_3\ .$$

Equations (A6) have nontrivial solutions if $\det\left((\partial\Gamma_j/\partial\rho_i)\right)=0$. Substituting (A4) into the last equality we obtain $\det\left((\partial\Gamma_j/\partial\rho_i)\right)=\Gamma_4=0$. In other words, $\mathsf{L}\left(\rho_1,\rho_2,\rho_3\right)$ attains its extremum at the planes $\rho_1=\rho_2$, $\rho_2=\rho_3$ and $\rho_3=\rho_1$ where

$$K_{4i}\left(\rho_1,\rho_2,\rho_2\right)\neq 0\ ,\quad K_{4i}\left(\rho_3,\rho_2,\rho_3\right)\neq 0\ ,\quad K_{4i}\left(\rho_1,\rho_1,\rho_3\right)\neq 0\ ,\qquad i=1,2,3\ .$$

A cyclic invariance (A3) of $\mathsf{L}\left(\rho_1,\rho_2,\rho_3\right)$ makes all three planes equivalent in the sense that provides for $\mathsf{L}\left(\rho_1,\rho_2,\rho_3\right)$ the same kind of extremum which can be only a minimum due to (A2). Consider one of the solutions, when $\rho_1\neq\rho_2=\rho_3$,

$$\mathsf{L}\left(\rho_1,\rho_2,\rho_2\right)=\frac{\left[(1+\rho_1)(1+\rho_2)^2+max\{1,\rho_1\rho_2^2\}\right]^2}{(1+\rho_1\rho_2+\rho_1)(1+\rho_1\rho_2+\rho_2)(1+\rho_2^2+\rho_2)}\ . \tag{A7}$$

This function possesses additional invariance under inversion of both variables,

$$\mathsf{L}\left(\rho_1,\rho_2,\rho_2\right)=\mathsf{L}\left(\frac{1}{\rho_1},\frac{1}{\rho_2},\frac{1}{\rho_2}\right)\ . \tag{A8}$$

Last relation (A8) simplifies essentialy further consideration since if in a region $\rho_1\rho_2^2\geq 1$ holds inequality $\mathsf{L}\left(\rho_1,\rho_2,\rho_2\right)>const$ then it also holds in a region $\rho_1\rho_2^2\leq 1$.

Consider a region $\rho_1\rho_2^2\geq 1$ and represent $\mathsf{L}\left(\rho_1,\rho_2,\rho_2\right)$ as follows,

$$\mathsf{L}\left(\rho_1,\rho_2,\rho_2\right)=3+\frac{\mathcal{U}^2-3\mathcal{W}}{\mathcal{W}}\ ,\quad \text{where} \tag{A9}$$

$$\mathcal{U}=(1+\rho_1)(1+\rho_2)^2+\rho_1\rho_2^2\ ,\quad \mathcal{W}=(1+\rho_1\rho_2+\rho_1)(1+\rho_1\rho_2+\rho_2)(1+\rho_2^2+\rho_2)\ .$$

Denoting $\rho_1=\varepsilon+1/\rho_2^2$, $\varepsilon\geq 0$, we get

$$\frac{\mathcal{U}^2-3\mathcal{W}}{\mathcal{W}}=\frac{1}{\rho_2}\,\frac{\mathcal{T}_0+\mathcal{T}_1\varepsilon+\mathcal{T}_2\varepsilon^2}{\mathcal{P}_0+\mathcal{P}_1\varepsilon+\mathcal{P}_2\varepsilon^2}\ ,\qquad \text{where} \tag{A10}$$

$$\mathcal{T}_0=(\rho_2-1)^2(1+\rho_2+\rho_2^2)^3\ ,\quad \mathcal{T}_1=\rho_2^2\left(1+(\rho_2-1)^2\right)(1+\rho_2+\rho_2^2)^2\ ,$$

$$\mathcal{T}_2=\rho_2^4\left(1+\rho_2(1+\rho_2)(1+\rho_2+\rho_2^2)\right)\ , \tag{A11}$$

$$\mathcal{P}_0=1+3\rho_2+6\rho_2^2+7\rho_2^3+6\rho_2^4+3\rho_2^5+\rho_2^6\ ,\quad \mathcal{P}_1=\rho_2^2\left(2+5\rho_2+8\rho_2^2+7\rho_2^3+4\rho_2^4+\rho_2^5\right)\ ,$$

$$\mathcal{P}_2=\rho_2^4\left(1+2\rho_2+2\rho_2^2+\rho_2^3\right)\ . \tag{A12}$$

One can see from (A11) and (A12) that $\mathcal{P}_0\geq 1,\mathcal{P}_1,\mathcal{P}_2,\mathcal{T}_0,\mathcal{T}_1,\mathcal{T}_2\geq 0$. Thus, $\mathsf{L}\left(\rho_1,\rho_2,\rho_2\right)$ arrives its minimal value 3 when $\varepsilon=0$, $\rho_2=1$. So the whole function $\mathsf{L}\left(\rho_1,\rho_2,\rho_3\right)$ also arrives its minimal value 3 when $\rho_1=\rho_2=\rho_3=1$.

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